Criticality in cellular automata

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CRITICALITY IN CELLULAR AUTOMATA

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Using recent results obtained for the transition to turbulence via spatiotemporal intermittency in extended dynamical systems, *critical* cellular automata rules are built. Thanks to a systematic procedure, the continuous phase transition observed in a coupled map lattice is translated into a sequence of cellular automata rules for which the critical properties of the original system are reproduced as precisely as desired. It is shown that criticality, as understood from the point of view of statistical mechanics, is intimately related to the various characteristics of Wolfram's class IV rules. This suggests in turn the picture of a "critical surface" in the space of rules and provides the basis for a discussion of the problem of classification schemes. We review recent cellular automata studies where questions linked to criticality arise and argue that they are unified in light of our results, leaving the relationship between computational and statistical characterizations of critical rules as the central problem for future studies on this subject.

1. Introduction

Whereas complexity is a notion often invoked in the field of cellular automata (CA) [1], "criticality" is much less commonly encountered, except perhaps in the study of the phase transitions of probabilistic cellular automata (PCA) such as directed percolation [2]. For deterministic cellular automata (DCA), partly due to their lack of continuous "control parameters", phase transitions and critical phenomena in the sense of statistical mechanics have only recently become topics of interest, along with the problem of the structure of the space of rules. This latter problem is obviously connected to the various attempts to define classifications of CA, either after their observed behavior (phenotypic classification) or after their rule table (genotypic classification). Indeed, once defined, classes of CA can be seen as phases, bringing up the question of the transitions between these phases, i.e. non-arbitrary paths in the space of rules that connect CA belonging to different classes.

It is well known that the traditional classification of Wolfram [3] suffers drawbacks, the most serious of which probably being its non-decidability [4]. It seems to us that the main problem is the exact status of class IV, and it may well be that its best definition remains founded on a negative statement: class IV rules are the rules which do *not* belong to any of the three other classes. Let us briefly recapitulate the (positive) definitions of these classes for infinite-size CA [3].

Class I automata evolve to a unique, homogeneous, stationary state after a finite transient from almost any initial condition.

Class II rules generate spatially inhomogeneous, stationary or periodic (with a short period), asymptotic states after a finite transient.

Class III automata evolve to a unique, chaotic, statistically well-defined state from almost any initial configuration in a finite number of timesteps. These qualitative definitions can be made quantitative using various statistical quantities.

Class IV was initially defined by various criteria (existence of propagative structures, arbitrary long transients, no smooth infinite volume limit, influence of low probability configurations on average quantities, etc.) but was mainly linked to the property of a rule to be a universal computer, i.e. of being able to perform any digital computation given a suitable initial configuration. But until now, the equivalence between this computational property and various statistical characterizations of class IV rules has remained an open question, mainly because proving that a rule is a universal computer is usually very difficult in the absence of any systematic approach.

Without a positive definition of class IV CA based on statistical properties, the completeness of Wolfram's classification cannot be ensured. Also, and particularly as defined above, the first three classes are not "closed": for example, it is practically impossible to decide membership in the case of very long transients.

Moreover, this phenotypic classification, together with other similar attempts to distinguish CA from their behavior [5,6], also suffers from its intrinsic inability to provide clues on the structure of the space of CA that would relate to the rule tables themselves. Ideally, one would like to at least be able to make an educated guess on the general behavior of a CA given some features of its rule table, and in particular see whether it is close to a transition point, possibly showing some critical dynamical properties.

In this paper, we take advantage of our recent work on the transition to spatiotemporal disorder in dynamical systems with many degrees of freedom to attack the above-mentioned problems, focusing on the possibility of critical behavior in CA and the relative importance of the corresponding models in the space of rules. Our point of view will often be that of statistical mechanics and we will mostly use the tools common in this domain of physics.

In a series of articles [7], we have shown that spatially extended dynamical systems such as partial differential equations, coupled ordinary differential equations, and coupled map lattices (CML), can exhibit transitions to disorder akin to phase transitions as defined in statistical mechanics. For all these models, these *transitions to turbulence via spatiotemporal intermittency* are observed when varying a continuous parameter, and the eventual critical points are well-characterized by means of the physical quantities similar to those used in the field of critical phenomena.

Here, after a brief presentation of this set of results, we focus on one particular such transition observed for a CML (section 2). Then, we connect those results to CA themselves, thanks to a systematic approximation scheme that enables us to construct CA rules with well-controlled critical properties (section 3). Section 4 is devoted to a general discussion of the relationships between various approaches to criticality in CA, with special attention to the aspects unveiled by our own results and their implications for the problem of the structure of the space of rules.

2. Critical behavior of a coupled map lattice

Before presenting the continuous phase transition observed in a CML that we will translate to a sequence of rules in the space of CA, let us briefly review the physical background that led us to this problem.

2.1. Transition to turbulence via spatiotemporal intermittency

Hydrodynamical turbulence is a phenomenon that may seem far away from the concerns usually at play in CA studies. Lattice gas models represent one way of connecting the two fields at a "microscopic" level. Recently, Pomeau suggested a higher level, macroscopic connection between some hydrodynamical situations and simple PCA such as directed percolation [8].

Among the flows most defiant of both analysis and experiments are those which exhibit the coexistence of a regular (laminar) state together with a disordered (turbulent) one. The corresponding dynamical regimes are characterized by patches of both states bordered by well-defined fronts evolving in space and time. Such *spatiotemporal intermittency* regimes are ubiquitous (e.g. turbulent spots in the Blasius boundary layer [9]), and Pomeau argued that their essential feature lies in the asymmetry between the two states. As a matter of fact, the intrinsic fluctuations of the disordered state allow the nucleation of the laminar state in a turbulent region, whereas the absence of such fluctuations in the regular state prevents the spontaneous emergence of disorder within a laminar patch. In the language of PCA, this is qualitatively equivalent to saying that the laminar state is *absorbing* and that the propagation of disorder can only happen through the contamination/recession process governing the dynamics of the local fronts separating two patches. It is then natural to propose a link between the relevant hydrodynamical situations and directed percolation which can be seen as one of the simplest two-state PCA with one absorbing phase.

Directed percolation is best known for exhibiting a continuous (second order) phase transition with rather well-understood critical properties in the infinite-size limit [2]. Such critical behavior is observed when varying a parameter governing the local transition probabilities defining the rule. Pursuing the analogy between PCA of the directed percolation type and systems showing spatiotemporal intermittency, one is led to imagine the occurence of similar scaling regimes in dynamical systems with many degrees of freedom when varying a control parameter through some criticial point.

This general picture was first confirmed by a study of a simple partial differential model believed to have some relevance to hydrodynamics [10]. We then introduced a simple CML designed to retain only the dynamical features necessary for exhibiting spatiotemporal intermittency [11]. This "minimal" model, to be presented below, was studied for space dimensions $1 \leq d \leq 4$ [11,12]. It was shown that the transition to turbulence via spatiotemporal intermittency can be of different types depending on the details of the system and the lattice dimension d. For $d \leq 2$, transitions are either discontinuous (first-order-like) or continuous (second-order-like) with possible complications due to the superposition of deterministic features on the general picture of a phase transition [13]. The equivalence with directed percolation suggested by Pomeau is thus not complete, but the conjecture is valid at a qualitative level, and remarkably anticipated the novel relevance of statistical mechanics in the field of spatially extended dynamical and physical systems brought to light by our findings.

In the next section, we exemplify these results by detailing a continuous phase transition observed in our minimal CML.

2.2. A typical continuous phase transition

A coupled map lattice consists of a collection of N identical local maps f of one or more real variables sitting on a lattice of space dimension d. The sites are updated synchronously by the iteration of the local map f and a coupling procedure involving the sites of a regular neighborhood. Most of the time, the evolution can be decomposed into two steps, iteration and coupling. This can be written as

$$\cdots \xrightarrow{g} X^n \xrightarrow{f} Y^n \xrightarrow{g} X^{n+1} \xrightarrow{f} Y^{n+1} \xrightarrow{g} \cdots$$

where g is the coupling function involving a site and its neighbors, and the superscripts symbolize the (discrete) time. Although we will briefly mention other situations, we restrict ourselves here to the most studied case of a chain ($d = 1$) of maps of one real variable ($X, Y \in \mathrm{R}$), coupled to their nearest neighbors by the linear "diffusive" coupling function g:

$$g(Y_{i-1}, Y_i, Y_{i+1}) = (1-\varepsilon)Y_i + \tfrac{1}{2}\varepsilon\,(Y_{i-1} + Y_{i+1}),$$

where ε is the coupling strength, and the subscripts denote the spatial indices of the sites.

The local map f designed to fulfill the minimal requirements for exhibiting spatiotemporal intermittency reads [11]:

$$\begin{aligned} f(X) &= sX && \text{if } X \in [0, 1/2], \\ f(X) &= s(1-X) && \text{if } X \in [1/2, 1], \\ f(X) &= X && \text{if } X > 1, \end{aligned}$$

with $s > 2$.

When uncoupled to its neighbors ($\varepsilon = 0$), each map eventually experiences a chaotic transient in the "turbulent" part of its phase space, $X \leq 1$, before reaching a fixed point in the laminar region,

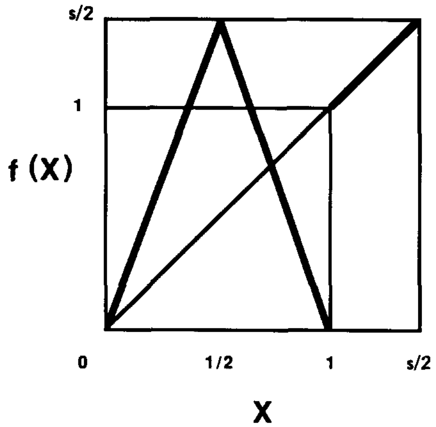

Fig. 1. Local map f of the minimal CML showing spatiotemporal intermittency. The turbulent state is the chaotic tent map ($X \leq 1$), and the laminar state is the continuum of marginally stable fixed points ($X > 1$).

$X > 1$ (see fig. 1). It is easy to show that the laminar state is absorbing in the coupled system. For strong enough ε and when starting from inhomogeneous initial conditions, the finite-amplitude perturbations introduced by the coupling may be large enough to take one quiescent site adjacent to an active one from the laminar state back to the turbulent state, and the CML then exhibits sustained regimes of spatiotemporal intermittency. There exists a threshold value ε_c (well-defined in the infinite size limit) above which this occurs, and the transition region shows scaling properties characteristic of continuous phase transitions (fig. 2).

The spatiotemporal intermittency regimes observed above threshold ($\varepsilon > \varepsilon_c$) possess well-defined statistical properties. For example, the average density of "turbulent" sites ($X > 1$), often considered as an order parameter, is the same whether the averaging is done in space, in time, or both. The following scaling properties are observed in the critical region:
– continuous decrease of the average density of turbulent sites when approaching the threshold from above (exponent β);
– divergence of the fluctuations of the instantaneous density of turbulent sites at threshold;
– divergence at threshold of the average transient time before a steady regime is reached;
– continuous decrease of the largest Lyapunov exponent when approaching threshold from above (preliminary result);
– algebraic distribution of the sizes and durations of clusters of laminar sites at threshold (exponents $\zeta_\perp$ and $\zeta_\parallel$);
– divergence of the coherence length $\xi_\perp$ and the coherence time $\xi_\parallel$ as defined from the characteristic scales extracted from the exponential distributions of sizes and durations of *laminar* clusters above threshold (fig. 3).

Most of these criteria will be used in the next section to characterize the critical CA rules emerging from our approximation of this CML. The scales mentioned in the last point are not the traditional quantities used to measure the coherence of a many-body system. Although this is still a point of ongoing controversy [14], let us mention here that autocorrelation functions do not provide clear-cut estimates of correlation scales for this system or for most systems of this type. The coherence scales defined above, on the other hand, are based on a reduction of the states of the CML deeply rooted in its dynamical properties and are relatively easy to interpret.

The equivalence with directed percolation initially proposed by Pomeau is not verified even in the case of a well-defined continuous transition. For example, the exponent $\zeta_\perp$ characterizing the distribution of sizes of laminar clusters at threshold takes two different values for $s = 2.1$ and $s = 3$ ($\zeta_\perp = 1.78$ and $\zeta_\perp = 1.99$) which are different from the corresponding value for directed percolation ($\zeta_\perp = 1.75$). This nonuniversality was part of our motivation to translate the problem into the field of DCA.

3. Sequences of CA rules approximating a continuous transition

In a previous work [15], we introduced an approximation of CML by CA in order to obtain insights on the various problems posed by the transition to turbulence via spatiotemporal intermittency and in particular the question of its nonuniversality. Here, we present a slightly different approximation scheme designed to produce CA as close as desired to a given critical point and study in some detail the sequences of rules equivalent to

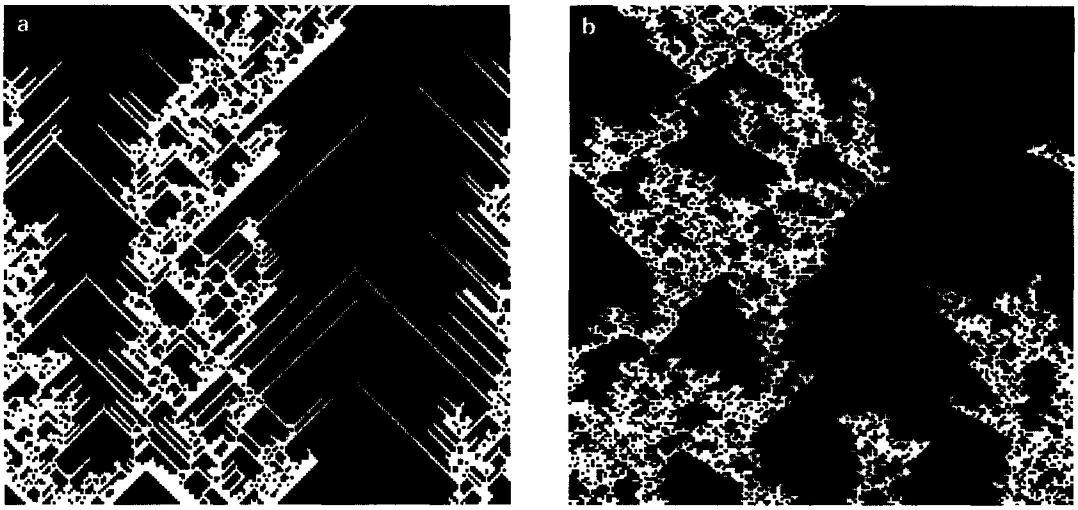

Fig. 2. Spatiotemporal intermittency in the critical regime for the minimal CML. Sites in the laminar (turbulent) state are black (white), following the natural binary reduction built in the local map f. The chain of $N = 200$ maps with periodic boundary conditions is shown just above the spatiotemporal intermittency threshold, and time is running upward. (a) $s = 3$, $\varepsilon = 0.360$, 200 iterations; (b) $s = 2.1$, $\varepsilon = 0.0047$, 200×16 iterations.

the continuous transition described in the previous section.

3.1. Approximating CML by CA

Only the discreteness of their local phase space distinguish DCA from CML. Indeed, a CML whose local map takes only discrete values (i.e. a step function) is equivalent to a DCA defined on the same lattice, the same neighborhood and a number of possible states equal to the number of steps of different heights in the step function (if the automaton is "observed" after the iteration step, cf. section 2.2). The approximation simply consists of choosing a step function $\tilde{f}$ preserving the essential features of the original local map f.

This very simple idea provides a systematic way of constructing sequences of rules approximating the variation of (continuous) parameters in the CML. Having an underlying "physical" signification, these sequences form non-arbitrary paths in the space of CA.

Virtually all parameters can be varied, including the space dimension and the size of the coupling neighborhood, so that the sequences of rules are not limited to a subset of the space of CA. For example, constructing CML based on the same local map f but with coupling functions g approximating the diffusion operator at higher orders (on larger and larger neighborhoods) creates a sequence of rules of increasing radius whose dynamics are very similar [16]. Thus, although obviously not a systematic way of exploring the structure of the space of CA, this type of approach produces well-controlled paths along all the dimensions of this space.

3.2. Previous results

In ref. [15], we constructed the step function $\tilde{f}$ approximating the local map f of our minimal CML by considering the preimages of the laminar state ($X > 1$) backward in time. This solution provided insights into the processes at play in the spatiotemporal intermittency regimes of the original CML. In particular, the cases $s = 2.1$ and $s = 3$ were found to produce very different sequences of rules when varying the coupling strength ε.

For $s = 2.1$, the rules of the sequence were all of class I and II, failing thus to reproduce the spatiotemporal intermittency regimes, whereas for $s = 3$, complex rules (class III and IV) were found above a threshold value ε_c depending on the order of the approximation. This indicated that

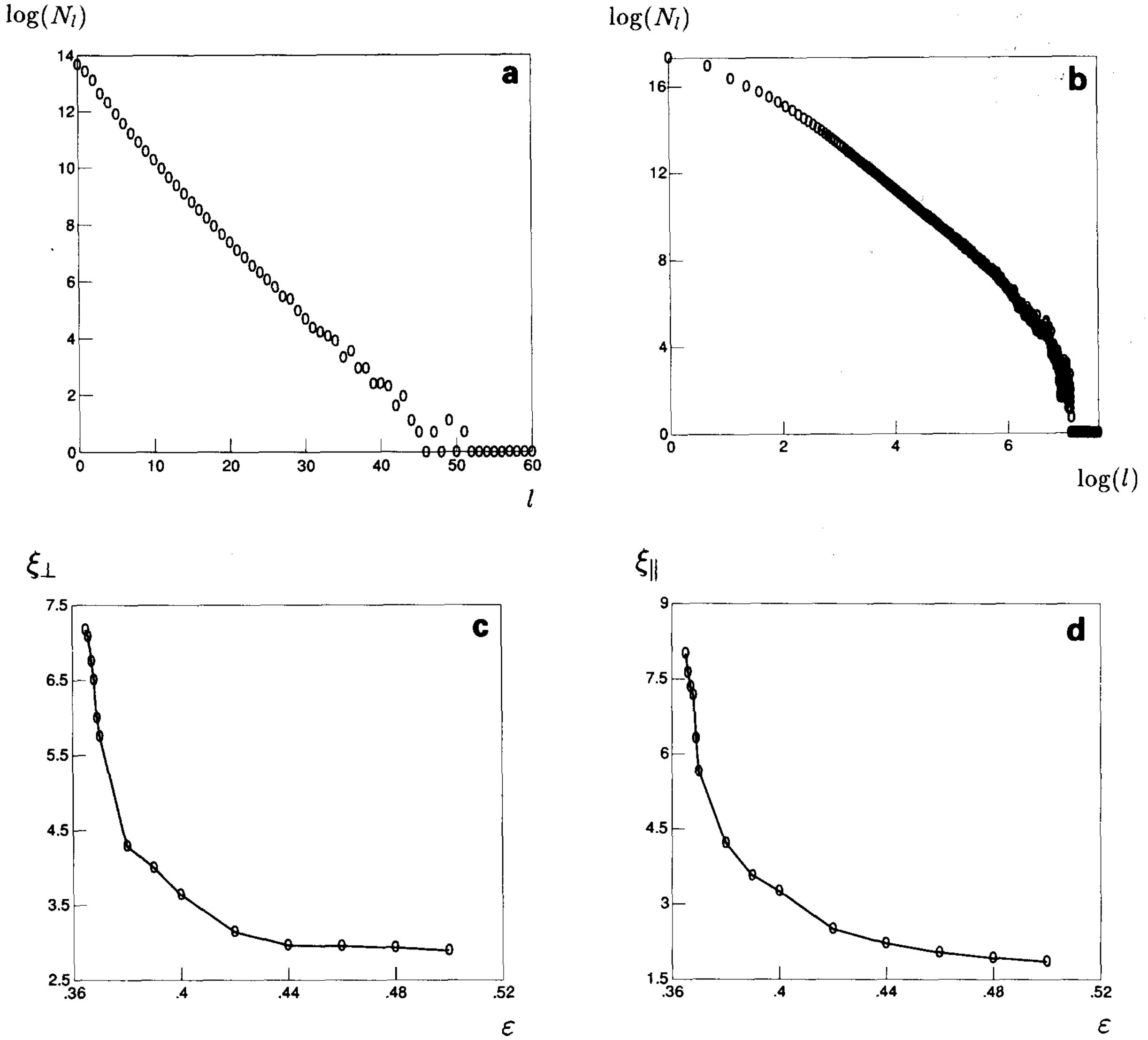

Fig. 3. (a) Histogram of the sizes of the clusters of laminar sites for the minimal CML above the spatiotemporal intermittency threshold ($s = 3$, $\varepsilon = 0.400 > \varepsilon_c = 0.360$). Statistics cumulated over space and time for a $N = 5000$ site chain with periodic boundary conditions during 5000 iterations. Exponential distribution with characteristic scale $\xi_\perp \sim 3.7$ sites. (b) Same as (a) but at the spatiotemporal intermittency threshold ($\varepsilon = 0.360$) for a lattice of 10000 sites during 10000 iterations. Algebraic distribution with critical exponent $\zeta_\perp$. (c) Variation with ε of the coherence length $\xi_\perp$ defined in (a). (d) Variation with ε of the coherence time $\xi_{||}$ defined in (a).

for $s = 2.1$ the local processes at the origin of the spatiotemporal disorder are due to the quasi-probabilistic mixing occuring during the long excursions of each site in the turbulent state. For $s = 3$, on the other hand, the spatiotemporal intermittency regimes are similar to those produced by class III and class IV rules, i.e. quasi-deterministic processes characterized by propagating structures and triangular clearings (fig. 2). The transitions, although both continuous for the CML, are very different, as already indicated by the nonuniversality of the critical exponents.

For $s = 2.1$, the critical regimes of the CML are close to those exhibited by a PCA rule interpolated between trivial (class I and II) DCA rules. Note that directed percolation falls into this category and indeed exhibits no particular structure in its spatiotemporal evolution, just as the minimal CML for $s = 2.1$. This may explain why the values of the critical indices of the transitions are close to each other for these two systems.

For $s = 3$, the transition is qualitatively different: the critical regimes of the CML are similar to those exhibited by class IV and class III rules, and one is led to imagine that there exist DCA rules as close as desired to this well-defined critical point.

The general picture of a *critical surface* in the space of rules emerges, a critical surface that can be reached either by interpolation between two rules lying apart from it (PCA) or by constructing DCA rules close to it. It is this latter possibility that interests us here, with the final aim of constructing critical CA models.

3.3. An equal-step approximation

The approximation of the local map f based on the preimages of the laminar state [15] creates step functions $\tilde{f}$ that do *not* converge to f when the order of the approximation increases. In order to ensure this convergence, we use here step functions $\tilde{f}_k$ ($k \geq 2$) which divide the turbulent part of the local phase space of f ($0 \leq X \leq 1$) in $2k-1$ steps of equal width having k different heights. The laminar part of the phase space ($1 < X \leq r/2$) is then also divided in steps whose heights match those previously defined. Fig. 4 shows a step function $\tilde{f}_3$ defined this way. The CML built on the step functions $\tilde{f}_k$ *is* a k-state DCA. For the one-dimensional case presented in section 2, the DCA have radius $r = 1$ (nearest-neighbor coupling in the CML) and their particular rule table depends on the parameters of the CML, s and ε.

The transition observed when varying continuously ε is translated into a discrete sequence of rules. The absorbing property of the laminar state is conserved in a global way: a configuration with all sites in the states corresponding to $X > 1$ will produce a site in one of those states. This justifies the continuing use of a binary representation (see fig. 2) even for the DCA with more than two states

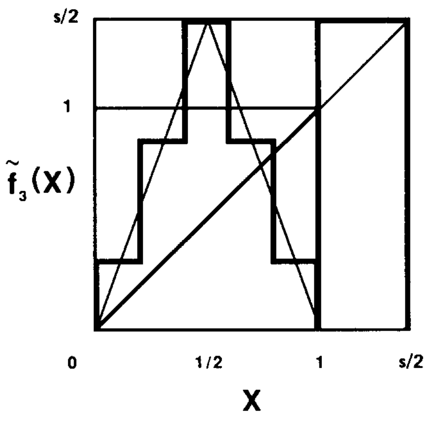

Fig. 4. Equal-width step function $\tilde{f}_3$ approximating the local map f of the minimal CML at order $k = 3$.

($k > 2$). This also allows a simple comparison with the original CML.

3.4. Results for the minimal CML with $s = 3$

Here we describe the sequences of rules produced by the approximation for the minimal CML defined in section 2.2 with $s = 3$ and ε varying from 0 to 1.

At first order ($k = 2$), the approximation is exactly equivalent to the one introduced in ref. [15]: independently of s ($2 < s \leq 3$), the sequence is composed of 7 of the 32 two-state three-neighbor legal rules studied in detail by Wolfram [17] (table 1).

Already at such a crude level, the transition to spatiotemporal intermittency observed for the CML is recovered, with a threshold ε_c separating class I and II rules from class III ones.

Increasing the order k of the approximation, more and more features of the original transition are recovered. Given the cardinal of the set of possible rules at fixed k, it is actually of no interest to define exactly the rules of the sequences for $k > 2$, so that we study them mostly through numerical simulations.

The number of rules in the sequence for $0 \leq \varepsilon \leq 1$ varies slightly with s but increases strongly with the order k of the approximation. Similarly, the mean normalized Hamming distance between consecutive rules in a given sequence decreases when k increases (fig. 5).

Table 1
Equivalent rules for the deterministic cellular automata approximating the minimal coupled map lattice at order $k = 2$ when the coupling ε is varied between 0 and 1 (results valid for $2 \leq s \leq 3$ only).

ε		$2/s - 4/s^2$		$1 - 2/s$		$4/s - 8/s^2$		$2 - 4/s$		$2/s$		$1 - 2/s + 4/s^2$	
Rule	32		36		4		76		94		90		122
Dynamics	trivial (class I and II)								complex (class III)				

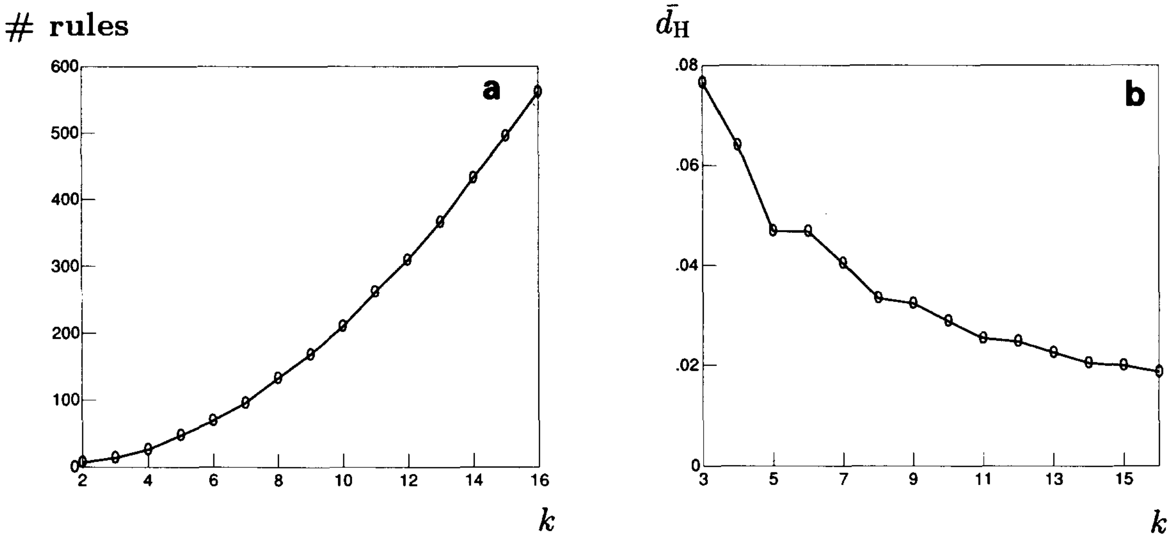

Fig. 5. (a) Variation with k of the number of rules in the sequences approximating the CML. (b) Variation with the order of the approximation k of the mean normalized Hamming distance $\bar{d}_H$ between rules in a sequence. Rules yielding different states for every parental neighborhood lie at distance 1 from each other after normalization.

This is not surprising since the phase space is then less and less coarse-grained, and the scheme converges to the original CML as $k \to \infty$. At a given order, there is a threshold value ε_c separating trivial from complex rules. Table 2 shows that this threshold rapidly converges to the corresponding value for the original CML. At high orders, say $k > 5$, the contiguity in the space of rules does imply a contiguity in the dynamics. This confirms the general remark that rules with similar rule tables are likely to exhibit similar behaviors, in relation with the observation that nevertheless certain configuration outputs can qualitatively modify the dynamics ("hot bits" [5]).

One of the most remarkable facts is that even at low orders of the approximation the rules in the transition region appear to have most of the properties of class IV rules. Fig. 6 shows such rules with characteristic propagating structures, very long transients, and seemingly unpredictable asymptotic states.

In the next section, we study the sequence generated by the approximation scheme at order $k = 10$ and compare it directly with the original transition to spatiotemporal intermittency in the minimal CML.

3.5. *The sequence of rules for $s = 3$ and $k = 10$*

At order $k = 10$, the transition to spatiotemporal intermittency of the minimal CML with $s = 3$ and $0 < \varepsilon < 1$ is reproduced by a sequence of 210 DCA rules. At this moderately high order, there are enough rules to be able to compare with the oiginal transition of the minimal CML presented in section 2.2.

The dynamics of these 210 rules show the existence of a threshold value $\varepsilon_c \simeq 0.39$ below which only class I and II rules are observed. The rules above the transition region possess the triangular clearings and the well-defined statistical prop-

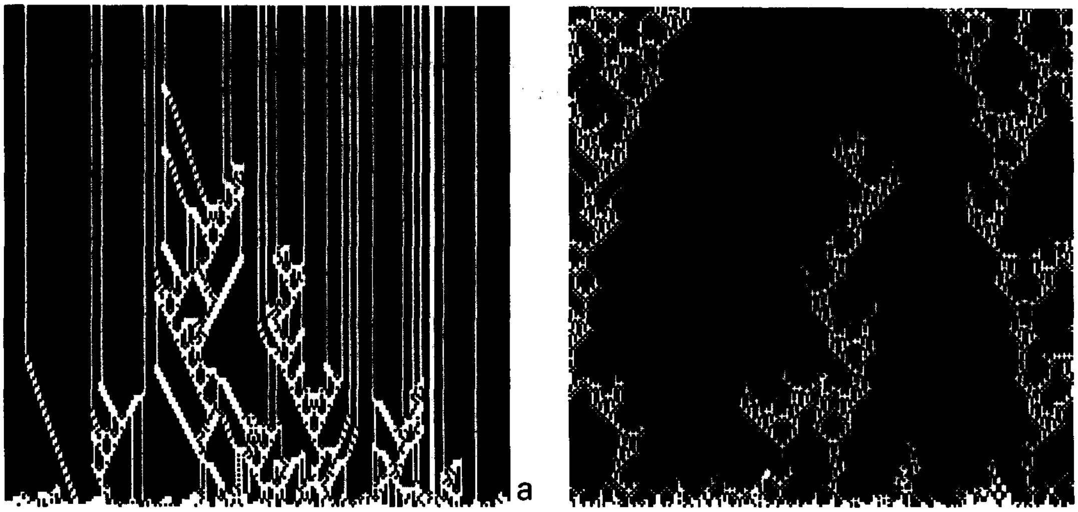

Fig. 6. Rules in the critical region of the sequences approximating the transition to turbulence via spatiotemporal intermittency in the minimal CML. Evolution of a lattice of 200 sites shown during 200 iterations under the binary reduction separating "turbulent" (white) from "laminar" (black) local states. Periodic boundary conditions, random initial conditions; time is running upward. (a) $k = 6$, $\varepsilon = 0.33$; (b) $k = 8$, $\varepsilon = 0.28$.

Table 2
Variation with order k of the threshold value ε_c separating trivial from complex rules in the sequences generated by the approximation. The parameter region comprising possibly critical rules is indicated within brackets. For higher orders of the approximation, there is no sharp limit defining the transition region. The spatiotemporal intermittency threshold for the minimal CML is $\varepsilon_c = 0.360$.

k	ε_c	Transition region
2	0.67	
3	0.50	
4	0.33	
5	0.50	[0.33–0.50]
6	0.42	[0.25–0.42]
7	0.472	[0.32–0.47]
8	0.434	[0.28–0.43]
9	0.42	[0.28–0.42]
10	0.39	[0.33–0.39]
11	0.376	
12	0.37	
13	0.364	

erties characteristic of class III CA. The transition region consists of highly complex, seemingly unpredictable rules. These rules exhibit irregular and sometimes very long transients when evolving from random initial conditions to asymptotic states, which usually depend on these initial conditions. Most of the features generally attributed to class IV rules are in fact observed. Fig. 7 shows the spatiotemporal evolution of four typical rules of the sequence.

A somewhat more quantitative comparison can be made with the continuous phase transition of the original CML. The characteristic properties discussed in section 2.2 are easily verified for the rules along the sequence, at least at a crude level, thanks to the natural binary reduction rooted in the dynamics of both the CML and the DCA produced by the approximation scheme. The triangular clearings of the class III rules appearing above threshold correspond to the laminar clusters of the spatiotemporal intermittency regimes of the CML. Indeed the corresponding states of the DCA are absorbing. The coherence scales $\xi_\perp$ and $\xi_\parallel$ defined in section 2.2 for the CML are equivalent to the characteristic length and time extracted from the exponential distribution of the sizes of triangular clearings for class III DCA evolving from random initial conditions [3].

This property of class III rules is conserved here when collapsing all absorbing states into a unique "laminar" state under the binary reduction. Fig. 8b shows the increase of $\xi_\perp$ for the rules of the sequence when approaching ε_c from above.

As for the critical exponents $\zeta_\perp$ and $\zeta_\parallel$ given by the algebraic distribution of the sizes of the lam-

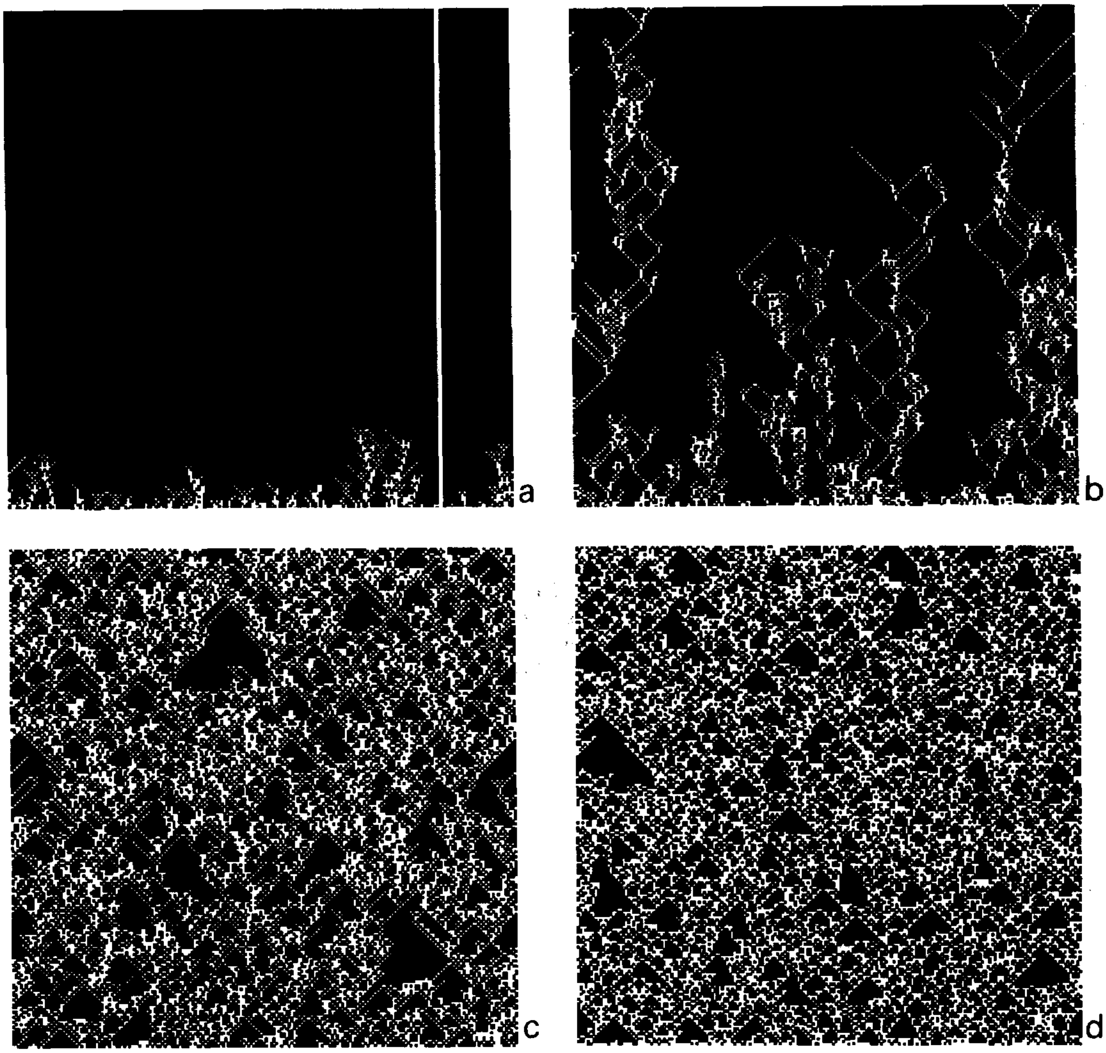

Fig. 7. Spatiotemporal evolution of four typical rules of the sequence generated by the approximation at order $k = 10$ for the transition of the minimal CML at $s = 3$. The lattice of 200 sites with periodic boundary conditions is shown under the natural binary reduction discussed in the text during the 200 iterations following random initial conditions. As in fig. 6, black sites are in one of the laminar (absorbing) states, white sites are "turbulent". (a) Class I rule well below threshold ($\varepsilon = 0.23$). (b) Rule slightly below threshold ($\varepsilon = 0.34$) (class IV ?) showing extremely long transients. (c) At threshold ($\varepsilon = 0.39$), first rule with a unique chaotic asymptotic state showing very long transients and quasi-algebraic distribution of laminar cluster sizes. (d) Class III rule above threshold ($\varepsilon = 0.50$). Note that the characteristic size of the triangular clearings is smaller far from threshold (d) than just above it (c).

inar clusters at threshold, it is not clear whether, increasing the order of the approximation, they "continuously" reach the corresponding value measured for the CML. Actually, as seen from fig. 8a, the rules in the transition region of the $k = 10$ sequence do not show a clear algebraic distribution of laminar cluster sizes, but rather lie in a crossover regime since exponential fits are not valid either. At higher orders of the approximation, the rules located around ε_c have a low mean density of "turbulent" sites, and the spatiotemporal patterns they develop locally resemble the fractal ones shown by class III rules evolving from simple seeds. For a DCA with a small number

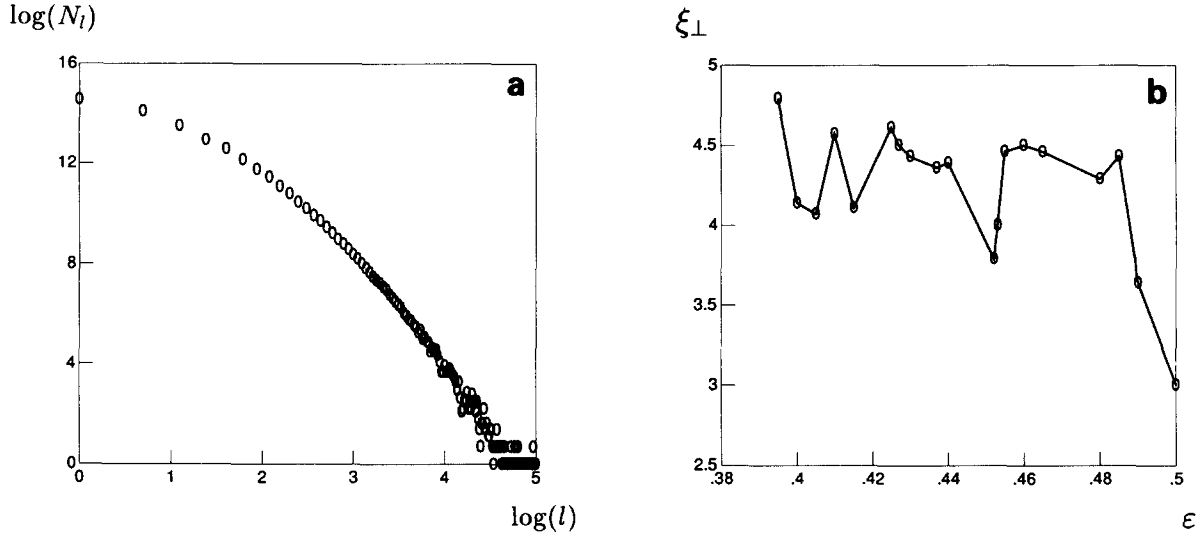

Fig. 8. (a) Histogram of the sizes of the "laminar" clusters for the rule of the sequence at order $k = 10$ and $\varepsilon = \varepsilon_c \simeq 0.39$ obtained on a lattice of 5000 sites during 5000 iterations following a long transient. (b) Variation along the sequence of the coherence length $\xi_\perp$ extracted from the exponential distribution of sizes of laminar clusters for the class III rules of the sequence at order $k = 10$ above threshold. Note that at this level of discretization, only the rough qualitative varation of $\xi_\perp$ is recovered, which indicates the still relatively important role of "hot bits" in the rule table.

of local states, there are but few possible corresponding fractal dimensions. But for the large-k limit of our approximation scheme, there are potentially many such dimensions. Since they are directly related to the critical exponents $\zeta_\perp$ and $\zeta_\parallel$ ($\zeta = 1 + d_f$), one is led to envision a continuum of possible critical indices in the $k \to \infty$ limit.

4. Discussion

4.1. Critical surface in the space of CA

The results given by our approximation of CML by CA provide insights into the general problem of the structure of the space of rules. In particular, the translation of the transition to turbulence via spatiotemporal intermittency (cf. section 2) into paths of DCA rules leads us to picture the existence of a critical surface on which lie "critical rules" defined by the usual criteria for critical points in statistical mechanics.

We have shown that the transition of the one-dimensional minimal CML defined in section 2.2 exhibits very different paths of rules under the approximation for $s = 2.1$ and $s = 3$. This indicates that there exist essentially two qualitatively distinct ways of approaching the critical surface and suggests that any rule close to it combines these two aspects in some proportion.

For $s = 2.1$, none of the rules produced by the approximation at low orders resides near the critical surface, and the critical regimes of the CML correspond to the probabilistic interpolation between the class I or class II rules appearing in the sequences, i.e. to a PCA of the type of directed percolation. The critical regimes of the CML can presumably be recovered only at very high orders, when the strong chaotic mixing occurring in the turbulent part of the phase space of the maps is reproduced by the complex interplay between the numerous local states of the DCA generated by the approximation. In this limit, these deterministic rules become equivalent to probabilistic ones, very much as the diadic map is related to a coin toss process [18].

In contrast, results from the approximation scheme in the $s = 3$ case pointed out that there are simple (small k) rules lying close enough to the critical surface to exhibit critical properties (over a range of scales depending on the distance to the critical point). In this case, the complexity observed in the spatiotemporal evolution arises from very elementary features of these rules rather than from the mixing of a quasi-continuum of local states.

A priori, we can conceive intermediate cases for which these two aspects could be present in any proportion. These cases could be generated for example by the approximation of the minimal CML for $2.1 < s < 3$, although the variation of s would not produce a continuous and monotonous variation in the properties of the critical rules (for example, ε_c does not vary continuously with s for the minimal CML [19]). Nevertheless, given the potential richness of large-k DCA, there may exist a continuous set of possible critical properties for the rules on the surface.

4.2. Classifications of CA

The results gathered also shed light on Wolfram's classification. Critical rules, if they are to be defined by the statistical criteria used here, appear as limit cases of class I, II and III CA. This is easily seen, for example, from the point of view of the mean duration of transients separating random initial conditions from an eventual asymptotic state. If this quantity or the corresponding r.m.s. diverge, as is the case when approaching the transition region under our approximation scheme (from below for class I and II, from above for class III), the rules can no longer be classified. The critical surface can thus be seen as the limit of class I, II and III rules having infinite transients, fluctuations, coherence scales, etc.

This conclusion also stems from the iterative parametric genotypic classification provided by local structure theory [20]. This mean-field type of analysis has to be pushed to higher and higher orders if one wants to account satisfactorily for the statistical properties of rules with increasing correlation scales. Eventually, the approach cannot be made precise enough for rules lying very close to the critical surface. The breakdown of local structure theory can thus be a clear signal of criticality. Incidentally, the coherence scales $\xi_\perp$ and $\xi_\parallel$ introduced here directly provide estimates of the order below which the analysis will fail. For truly critical rules, other methods should be considered, such as a renormalization-group treatment, which precisely tries to take advantage of the scaling properties [21].

4.3. Criticality and computational properties

If critical rules are defined as limit cases of class I, II or III CA, a central question is still the relationship between the existence of a critical surface and the universal computer property often used to characterize class IV rules. There is no systematic way of attacking this problem, but some comments already stated elsewhere [22,23] can be made. Critical rules may be characterized by their ability to undergo arbitrarily long transients. This is equivalent to stating the unpredictability of these rules. The undecidability of the finiteness of transients is also equivalent to the problem of the halting of a Turing machine.

Under our approximation scheme, critical rules often lie at the border between class II and class III CA. Following Wootters and Langton [23], this can be expressed in terms of the minimal requirements for a system to be able to perform calculations. Class III rules have good "communication" properties between sites (as seen from their positive Lyapunov exponents [17]) but almost no ability to store information (they reach a unique chaotic asymptotic state for almost all initial conditions). Class II rules, on the other hand, possess very good storage properties, but no communication is possible after they reach their spatially complex temporally trivial asymptotic state. In this context, critical rules seem to fulfill the necessary compromise between storage and communication at the root of any complex digital calculation.

4.4. Criticality and marginal stability

Recently, a number of rather simple CA models have been introduced and shown to exhibit scaling properties "spontaneously" [24]. We shall not discuss here their relevance to the physical phenomena they are taken as models of, but merely make the observation that these models of *self-organized criticality* may be seen as critical rules from our point of view. Given the picture of the critical surface of rules developed here, it is not surprising that such models exist, so that the main question is why they sit close to the critical surface.

A key concept for self-organized criticality is *marginal stability*. Typically, a system is marginally stable when in the state for which the propagation of a local perturbation is just able to go to infinity in time and in space. How does the notion of marginal stability relate to the critical CA rules constructed here?

In the context of the continuous transition to turbulence via spatiotemporal intermittency, the transition point can be seen as the threshold in parameter space beyond which disorder will propagate to infinity in space and time (cf. the analogy with directed percolation [7]). The critical regimes of spatiotemporal intermittency can thus be seen as marginally stable states. Moreover, for the minimal CML presented in section 2, the continuous character of the transition (i.e. its criticality) seems to be intimately related to the fact that the laminar region is made of a continuum of marginally stable fixed points. Indeed, in the case of a unique stable fixed point, the transition is discontinuous [13]. Criticality is linked to the possibility for an infinitesimal local perturbation to bring a site from the laminar to the turbulent state and thus allow propagation of disorder at a negligible cost. Marginal stability appears to be an essential ingredient for an extended system to undergo a continuous transition to spatiotemporal intermittency. Coming back to the approximation of CML by CA presented here, it is clear that marginal stability is also inscribed in the critical rules generated in the transition region.

This point of view has also been recently discussed by Gutowitz [25] and McIntosh [26] at the level of mean-field theory. In this context, class I and class II rules can easily, although not strictly, be related to the existence of a unique stable fixed point at the origin of their mean-field map, whereas class III rules are characterized by the existence of a non-trivial stable fixed point. Critical rules are then naturally conjectured to represent, here again, the intermediate cases, among which the presence of a marginally stable fixed point for the mean-field map is generic.

Although mean-field theory in known to be unable to account for the details of the statistical properties of CA, it is usually believed to provide good estimates of certain important quantities [27]. This justifies the above conjecture, but it is not an absolute criterion to decide whether a rule will show critical properties. The indications given by the mean-field analysis are never quantitative (for example, the mean-field map overestimates the mean densities). Moreover, their use for critical rules seems particularly daring since these rules are characterized by their long range correlations whereas the mean-field analysis consists primarily of neglecting them. Nevertheless, the work of McIntosh does bring evidence that a marginally stable fixed point in the mean-field map is a good indication of critical behaviors for CA.

5. Conclusion

The approximation scheme first introduced in [15] and presented here in a slightly different version has permitted the construction of CA rules whose critical properties are well controlled. Although not a tool for a systematic and comprehensive exploration of the space of CA, it allows the design of sequences of rules mimicking phase transitions occurring in extended dynamical systems and ensures the existence of critical rules, pictured to lie on a critical surface.

From the results, a certain unity in the various statistical characterizations of Wolfram's class IV, now seen as a limit case of the other classes, is recovered.

A central problem remains, then, which is the nature of the relationship between the statistical characterizations of critical rules and the computational properties discussed in section 1. When does a rule with scaling properties have the property of being a universal computer? Conversely, how critical is such a rule? These are the key questions for a further progress in understanding criticality in cellular automata.

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